

MOMENT MAPS AND NON-COMPACT COBORDISMS

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ABSTRACT. We define a moment map for a torus action on a smooth manifold *without* a two-form. Cobordisms of such structures are meaningful even if the manifolds are non compact, as long as the moment maps are proper. We prove that a compact manifold with a torus action and a moment map is cobordant to the normal bundle of its fixed point set. Two formulas follow easily: Guillemin's topological version of the abelian Jeffrey-Kirwan localization, and the Guillemin-Lerman-Sternberg formula for the Duistermaat-Heckman measure.

1. INTRODUCTION

This work is part of a joint project with Victor Guillemin and Viktor Ginzburg, in which we use cobordisms to study group actions on symplectic manifolds. Some of our results were announced and their proof sketched in [GGK1].

In this paper we introduce a new, cleaner, cobordism technique. We treat non-compact manifolds directly, without “approximating” by compact manifolds as we did in [GGK1]. Our basic theorem involves manifolds and not orbifolds; orbifolds only come up as reduced spaces. And we don't need to carry along a symplectic (or presymplectic) form for the theory to work.

Our basic object is a manifold with a torus action, a proper moment map, and an equivariant cohomology class. If (M, ω, Φ) is a symplectic manifold with a moment map in the usual sense, one can take the equivariant cohomology class represented by the sum $\omega + \Phi$. However, in our new approach we define a moment map without a two-form (see section 2), and our equivariant cohomology class is unrelated to the moment map. The normal bundle of every connected component of the fixed point set is a (non-compact) manifold with the same structure: a torus action, a proper moment map (that needs to be chosen; see section 5), and an equivariant cohomology class. We prove in section 6 that the original manifold, if compact, is cobordant to the disjoint union of these normal bundles. The cobording manifold carries the same kind of structure, and all moment maps are proper. In particular, if the torus acted with isolated fixed points, the manifold is cobordant to a disjoint union of vector spaces.

The motivation for this work comes from the formula of Guillemin-Lerman-Sternberg [GLS1]. This formula expresses the Duistermaat-Heckman measure for a compact symplectic manifold with a torus action with isolated fixed points as the sum of Duistermaat-Heckman measures for the tangent spaces at the fixed points. Viktor Ginzburg observed several years ago [G1] that the Duistermaat-Heckman measure is an invariant of (compact) cobordism (in the sense of [G2]). This tempts one to assert that the manifold is cobordant to the disjoint union of its tangent

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spaces at the fixed points. In sections 2–6 we make sense of this assertion. In section 6 we prove it. In sections 7 and 8 we state and prove that “cobordism commutes with reduction”, a phenomenon first shown and used by Guillemin in the context of symplectic manifolds (see [GGK1]). We use this in sections 9, 10, and 11 to deduce (generalizations of) the Guillemin-Jeffrey-Kirwan formula and the Guillemin-Lerman-Sternberg formula.

Another application of our cobordism technique, that will appear elsewhere (at least in [GGK2]), is a new proof of the Atiyah-Bott-Berline-Vergne localization for equivariant differential forms. This uses the technique of Lerman cutting applied to spaces with proper moment maps.

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2. MOMENT MAPS

Let T be a torus and \mathfrak{t} its Lie algebra. If H is a closed subgroup of T and \mathfrak{h} is the Lie algebra of H , the H -component of a map into \mathfrak{t}^* is the composition of the map with the natural projection $\mathfrak{t}^* \rightarrow \mathfrak{h}^*$.

Definition 2.1. Let M a smooth manifold with a smooth T -action. A *moment map* is a smooth map $\Phi : M \rightarrow \mathfrak{t}^*$ with the following properties:

1. Φ is T -invariant
2. For any subgroup H of T , the H -component of Φ is constant on each connected component of the fixed point set of H in M .

Remark. In property 2 it is enough to take circle subgroups H of T .

Example 2.2. A constant function is a moment map.

Remark. A linear combination of moment maps is a moment map. Later on we will demand that our moment maps be *proper*. Linear combinations of proper moment maps might no longer be proper moment maps. However, if $\eta \in \mathfrak{t}$ is a vector and Φ_1 and Φ_2 are two moment maps whose η -components, $\langle \Phi_i, \eta \rangle$, are proper and bounded from below, then so is every positive combination of Φ_1 and Φ_2 .

Example 2.3. If T is a circle, and we identify \mathfrak{t}^* with \mathbb{R} , a moment map is any real valued function that is constant on each connected component of the set of fixed points.

Example 2.4. On a vector space on which the circle acts linearly and fixes only the origin, the function $\Phi(v) = \|v\|^2$, with respect to any invariant metric, is a moment map that is proper and bounded from below.

Example 2.5. If ξ_M is a vector field on M that generates a circle action, the function $\Phi(x) = \langle \xi_x, \xi_x \rangle$, where \langle, \rangle is any invariant Riemannian metric, is a moment map that is proper and bounded from below.

Example 2.6. Suppose (M, ω) is symplectic and Φ is a moment map in the traditional sense, i.e., $\iota(\xi_M)\omega = -d\langle\Phi, \xi\rangle$ for all $\xi \in \mathfrak{t}$, where ξ_M is the vector field on M that generates the action of the one parameter subgroup $\exp(s\xi)$, $s \in \mathbb{R}$. Then Φ is a moment map in the sense of Definition 2.1.

3. EQUIVARIANT COHOMOLOGY

We will work with triples (M, Φ, c) where M is an oriented manifold acted upon by a torus T , where $\Phi : M \rightarrow \mathfrak{t}^*$ is a proper moment map, and where c is an equivariant cohomology class¹ on M (see below). We keep track of the torus action on M although it is not explicit in our notation.

Recall, an equivariant differential form is a polynomial on the Lie algebra \mathfrak{t} whose values are differential forms on M . The equivariant exterior derivative is defined by $(d_T\alpha)(\xi) = d(\alpha(\xi)) + \iota(\xi_M)\alpha(\xi)$ for all $\xi \in \mathfrak{t}$, ξ_M being the corresponding vector field on M . The equivariant cohomology (over \mathbb{R}) of a manifold is the quotient $\ker(d_T)/\text{image}(d_T)$. Good references on equivariant cohomology and equivariant differential forms are [AB] and [BGV].

For example, if (M, ω) is a symplectic manifold with a torus action and Φ is a moment map, the sum $\omega + \Phi$ is a closed equivariant form. However, an important point is that *one can work with equivariant forms that have nothing to do with the moment map*.

4. COBORDISM

For cobordism to be meaningful one must make some compactness assumption. Otherwise, any manifold M would be cobordant to zero via the noncompact cobordism $M \times (0, 1]$. However, we can drop the compactness assumption if we demand the cobordism to carry a proper moment map:

Definition 4.1. Let M, M' be smooth oriented manifolds with actions of a torus T , let $\Phi : M \rightarrow \mathfrak{t}^*$ and $\Phi' : M' \rightarrow \mathfrak{t}^*$ be proper moment maps, and let c and c' be equivariant cohomology classes on M and M' respectively. A *cobordism* between the triples (M, Φ, c) and (M', Φ', c') is a triple $(W, \tilde{\Phi}, \tilde{c})$, where W is an oriented manifold-with-boundary with a T -action, $\tilde{\Phi} : W \rightarrow \mathfrak{t}^*$ is a proper moment map, and \tilde{c} is an equivariant cohomology class on W , and an equivariant diffeomorphism of the boundary ∂W with the disjoint union $M \sqcup M'$ that carries $\tilde{\Phi}$ to $\Phi \sqcup \Phi'$, carries \tilde{c} to $c \sqcup c'$, and carries the boundary orientation on ∂W to the given orientation of M and the opposite orientation on M' .

Example 4.2. Every ordinary cobordism of compact oriented manifolds is also a cobordism in the sense of Definition 4.1 when we take the trivial T -actions, zero moment maps, and zero cohomology classes.

Example 4.3. Suppose that M is a manifold with a circle action and that $\Phi_i : M \rightarrow \mathbb{R}$, $i = 0, 1$, are moment maps that are proper and bounded from below and that take the same values on the fixed points. Then (M, Φ_0) is cobordant to (M, Φ_1) ; take the cobording manifold $M \times [0, 1]$ with the moment map $(m, t) \mapsto (1 - t)\Phi_0(m) + t\Phi_1(m)$.

¹ our cohomology is always over \mathbb{R}

Example 4.4. A space with a trivial torus action and a proper moment map must be compact, because its moment map must be constant. Two such spaces are cobordant in the sense of Definition 4.1 if and only if they are cobordant in the usual (compact) sense. This is because on a connected manifold-with-boundary, if a torus action acts trivially on the boundary (or, in fact, on any submanifold of codimension one), it acts trivially everywhere.

Remark. If two compact manifolds with torus actions and proper moment maps are cobordant in the sense of Definition 4.1 via a non-compact cobordism, they are also cobordant in the sense of Definition 4.1 via a compact cobordism. Thus our notion of cobordism refines the usual notion of equivariant cobordism for compact manifolds with torus actions. To show this, first use Lerman’s cutting procedure [L] to replace a noncompact cobordism by a compact orbifold cobordism, then get rid of the orbifold singularities by successive blowups along singular strata (with respect to a complex structure on the normal bundles to the strata). These blowups are T -equivariant, and the cohomology class pulls back via the blowups. Details will appear in [GGK2]. In this paper we do not use this result.

Remark. The above Examples show that our notion of cobordism is non-trivial. Note that it is absolutely crucial for the moment maps to be proper; otherwise, every manifold M would be cobordant to the empty set via the non-compact cobordism $M \times (0, 1]$, where the T action and moment maps are induced from those on M .

Also notice that it is crucial to assume Condition 2 in Definition 2.1 of the moment map. If we drop this condition and only work with, say, manifolds equipped with proper functions to \mathbb{R} , any compact manifold M would be cobordant to the empty set via the non-compact cobordism $M \times (0, 1]$ and the proper function $\tilde{\Phi}(m, t) = \Phi(m) + 1/t - 1$.

Lemma 4.5. *Cobordism is an equivalence relation.*

Proof. For smooth manifolds with no extra structure, this is a standard fact in differential topology. See, e.g., [BJ]. It is not hard to adapt the standard proof to manifolds equipped with a T -action, a moment map, and an equivariant cohomology class. We omit the details; we might include them in [GGK2]. \square

5. LINEAR ACTIONS

The most useful non-compact manifolds are vector spaces.

Recall that a complex representation of the torus T is determined up to isomorphism by its *weights*, which are elements of ℓ^* , the integral weight lattice in \mathfrak{t}^* . On the underlying real representation, the weights are defined only up to sign; we call these elements of $\ell^*/\pm 1$ *real weights*. A representation of a torus on a real vector space V is determined up to isomorphism by the dimension of the subspace, F , of fixed vectors, and by the real weights for the torus action on the quotient V/F .

Fix a real representation of T . Let $\{\pm\alpha_i\}$ be its nonzero real weights. Choose a “polarizing” vector $\eta \in \mathfrak{t}$, a vector such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i . The corresponding *polarized weights* are the elements $\alpha_i^\eta \in \{\alpha_i, -\alpha_i\}$ of ℓ^* with the sign chosen so that $\langle \alpha_i^\eta, \eta \rangle > 0$.

The following Lemma shows that a moment map for a linear torus action is determined up to cobordism by its value at the origin and its behavior at infinity, provided that some component of the moment map is proper.

Lemma 5.1. *Let V be a real vector space with a linear action of the torus T that fixes only the origin. Let $\pm\alpha_1, \dots, \pm\alpha_m$ be the real weights for the action (without repetitions). Let $V = V_1 \oplus \dots \oplus V_m$ be the decomposition of V into invariant subspaces such that T acts on V_i by the real weight $\pm\alpha_i$. Any vector $v \in V$ can be expressed uniquely as a sum, $v = \sum v_i$, where v_i is in V_i . Choose a vector $\eta \in \mathfrak{t}$ such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i . Let $\alpha_1^\eta, \dots, \alpha_m^\eta$ be the corresponding polarized weights. Choose any invariant metric on V . Let a be any element of \mathfrak{t}^* . Then*

$$\Phi(v) = a + \sum_{i=1}^m \|v_i\|^2 \alpha_i^\eta$$

defines a moment map on V whose value at the origin is a and whose η -component, $\langle \Phi, \eta \rangle$, is proper and bounded from below. Any other moment map, Φ' , with these properties is cobordant to Φ , i.e., the triple (V, Φ', c) is cobordant to the triple (V, Φ, c) for any equivariant cohomology class c on V .

Proof. Let $\Phi' : V \rightarrow \mathfrak{t}^*$ be another moment map whose value at the origin, $\Phi'(0)$, is a , and whose η -component, $\langle \Phi', \eta \rangle$, is proper and bounded from below. Let I be the closed interval $[0, 1]$. On the product $I \times V$, take the T -action induced from that on V . The function $\tilde{\Phi}(t, v) := (1-t)\Phi(v) + t\Phi'(v)$ is proper because its η -component is proper. It is a moment map because the fixed point set of any subgroup of the torus contains the origin, 0, and because $\Phi(0) = \Phi'(0)$. The pair $(I \times V, \tilde{\Phi})$ provides a cobordism between (V, Φ) and (V, Φ') . An equivariant cohomology class c on V can be simply pulled back to $I \times V$. \square

A similar result holds for vector bundles:

Lemma 5.2. *Let E be a real vector bundle over a compact connected manifold F . Suppose that the torus T acts linearly on the fibers of E , fixing only the zero section. Let $\pm\alpha_i \in \mathfrak{t}^*$ be the real weights for this action. Let $E = E_1 \oplus \dots \oplus E_m$ be the decomposition of E into sub-vector-bundles such that the torus T acts on the fibers of E_i by the real weight $\pm\alpha_i$. Fix a vector $\eta \in \mathfrak{t}$ such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i . Any element v in the vector bundle E can be expressed uniquely as a sum, $v = \sum v_i$, where v_i is in E_i . Fix an invariant inner product on the fibers of E . Let a be any element of \mathfrak{t}^* . Then*

$$\Phi(v) = a + \sum_{i=1}^n \|v_i\|^2 \alpha_i^\eta$$

defines a moment map on E whose value on the zero section is a and whose η -component is proper and bounded from below. Any other moment map with these properties is cobordant to Φ .

Proof. The proof is completely analogous to the proof of Lemma 5.1: if $\Phi' : E \rightarrow \mathfrak{t}^*$ is a moment map that takes the value a on the zero section and whose η -component is proper and bounded from below, the product $I \times E$ with the moment map $\tilde{\Phi}(t, v) = (1-t)\Phi(v) + t\Phi'(v)$ provides the desired cobordism. \square

Moreover,

Lemma 5.3. *Let $\pi : E \rightarrow F$ be a real vector bundle over a compact connected manifold. Suppose that the torus T acts on E by bundle maps and that $H \subset T$ is a subgroup whose fixed point set is exactly the zero section. Let $\pm\alpha_i \in \mathfrak{h}^*$ be the real weights for the action of the identity component, H_0 , of H on the fibers of E . Fix a*

vector $\eta \in \mathfrak{h}$ such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i . Let $\alpha_i^\eta = \pm \alpha_i$ be the polarized weights, with $\langle \alpha_i^\eta, \eta \rangle > 0$. For $v \in E$, let $v = \sum v_i$ be the decomposition such that H_0 acts on v_i with weight α_i . Let

$$\Phi_1(v) = \sum_{i=1}^n \|v_i\|^2 \alpha_i^\eta$$

be the H -moment map on the fibers of E as described in the previous lemma. For every embedding $i : \mathfrak{h}^* \hookrightarrow \mathfrak{t}^*$ whose composition with the restriction $\mathfrak{t}^* \rightarrow \mathfrak{h}^*$ is the identity, the composition $i \circ \Phi_1 : E \rightarrow \mathfrak{t}^*$ is a T -moment map.

Let $\Phi_2 : F \rightarrow \mathfrak{t}^*$ be any T -moment map on the base space, F , identified with the zero section of E . (Since H acts trivially and F is connected, the image of Φ_2 is contained in a translation of the annihilator, \mathfrak{h}^0 , of \mathfrak{h} in \mathfrak{t}^* .) The composition $\Phi_2 \circ \pi : E \rightarrow \mathfrak{t}^*$ is also a T -moment map, because the projection π is T -equivariant.

The sum $i \circ \Phi_1 + \Phi_2 \circ \pi$ is a T -moment map on E whose η -component is proper and bounded from below. Any other T -moment map on E whose η -component is proper and bounded from below and whose restriction to the zero section is Φ_2 is cobordant to this sum.

Proof. Again, if $\Phi' : E \rightarrow \mathfrak{t}^*$ is such a moment map, take $I \times E$ with the moment map $\tilde{\Phi}(t, v) = (1-t)\Phi(v) + t\Phi'(v)$ where $\Phi = i \circ \Phi_1 + \Phi_2 \circ \pi$. \square

6. T -FIXED POINTS

We now state our basic theorem.

Theorem 1. *Let M be an oriented manifold with an action of a torus T , let Φ be a moment map, and let c be an equivariant cohomology class on M . Suppose that the vector $\eta \in \mathfrak{t}$ is such that the η -component of the moment map,*

$$\langle \Phi, \eta \rangle : M \rightarrow \mathbb{R},$$

is proper and bounded from below. Denote by M^η the zero set of the vector field on M corresponding to η . Every connected component, F , of M^η is a compact T -invariant submanifold of M . Let NF be the normal bundle of F in M with the T -action induced from that on M . Let c_F be the pullback of the cohomology class c via the maps $NF \rightarrow F \hookrightarrow M$. Let Φ_F^η be a moment map on NF defined by

$$\Phi_F^\eta(v) = \Phi(p) + \sum_i \|v_i\|^2 \alpha_{i,F}^\eta \quad v \in N_p F \quad (6.1)$$

as in the notation Lemma 5.3 with $E = NF$. Then the triple (M, Φ, c) is cobordant to the disjoint union, over all the components F of the set M^η , of the triples (NF, Φ_F^η, c_F) .

An important special case is that of a torus action with isolated fixed points on a compact manifold:

Theorem 2. *Let M be a compact oriented manifold with an action of a torus T with isolated fixed points, let Φ be a moment map, and let c be an equivariant cohomology class on M . Let $\eta \in \mathfrak{t}$ be any nonzero vector that does not belong to the Lie algebra of the stabilizer of any point in M . For every fixed point, p , in M , let T act on the tangent space $T_p M$ by the linear isotropy T -action, and let c_p be the pullback of the equivariant cohomology class c via the maps $T_p M \rightarrow \{p\} \hookrightarrow M$. Let Φ_p^η be a moment map on NF defined as in Lemma 5.1 with $V = T_p M$. Then the*

triple (M, Φ, c) is cobordant to the disjoint union of the triples $(T_p M, \Phi_p^\eta, c_p)$ over all fixed points p in M .

Proof of a special case of Theorem 1. To convey the idea of the proof let us first deal with a circle action with isolated fixed points. Let $\Phi : M \rightarrow \mathbb{R}$ be a moment map that is proper and bounded from below. We would like to show that the pair (M, Φ) is cobordant to the union of the pairs $(T_p M, \Phi_p)$ over the fixed points, p , of the circle action, where $\Phi_p : T_p M \rightarrow \mathbb{R}$ is defined by $\Phi_p(v) = \Phi(p) + \|v\|^2$ for some invariant metric on $T_p M$.

Take the product of M with the half-open interval:

$$M \times (0, 1],$$

with the circle action induced from that on M . This provides a non-compact cobordism of M with the empty set. The function $(m, x) \mapsto \Phi(m)$ is a moment map whose restriction to the boundary, $M \times \{1\}$, is Φ . However, this moment map is not proper. We fix this by adding a function $\rho(x)$, $0 < x \leq 1$, that approaches ∞ as x approaches 0, and that vanishes for $\epsilon \leq x \leq 1$. If there are no fixed points, the function

$$\tilde{\Phi}(m, x) = \Phi(m) + \rho(x)$$

is a proper moment map whose restriction to the boundary, $M \times \{1\}$, is Φ , and we are done.

If there are fixed points, this does not work: the components of the fixed point set in $M \times (0, 1]$ have the form $F \times (0, 1]$ where F is a component of the fixed point set in M . The function $\tilde{\Phi}$ is constant on $F \times [\epsilon, 1]$ but not on $F \times (0, \epsilon]$, so it is not constant on the fixed components, so it is not a moment map. However, in this case we can take the same function $\tilde{\Phi}$ on the manifold

$$(M \times (0, 1]) \setminus \sqcup_p B_p,$$

where B_p is an ϵ -neighborhood of $(p, 0)$ in $M \times (0, 1]$ with respect to some invariant metric on M . The union is over the (isolated) fixed points p in M . Here ϵ is the same as in the definition of $\tilde{\Phi}$, and the metric on M is chosen so that ϵ -neighborhoods of fixed points are disjoint from each other, and so that an ϵ -neighborhood of the fixed point p is equivariantly diffeomorphic to $T_p M$. The removal of B_p creates a boundary component that is equivariantly diffeomorphic to a ball in M around p via the projection map $(m, x) \mapsto m$. This is, further, equivariantly diffeomorphic to $T_p M$. The function $\tilde{\Phi}$ transforms to a function $\Phi_p(v)$ on $T_p M$ that takes the value $\Phi(p)$ at the origin and that approaches ∞ as we approach infinity in $T_p M$. By Lemma 5.1, this function is cobordant to Φ_p . \square

Proof of Theorem 1. We return to the full generality of torus actions. Let $H \subseteq T$ be the closure of the one parameter subgroup $\{\exp(s\eta), s \in \mathbb{R}\}$ of T . Since the set M^η coincides with the fixed point set, M^H , of H in T , it consists of a disjoint union of submanifolds of M .

Let F be a connected component of the set M^η . Since the η -component of Φ is proper and is constant on F , F is compact. There exists an invariant Riemannian metric on M and an $\epsilon > 0$ such that the ϵ -neighborhood of F in M is equivariantly diffeomorphic to the normal bundle, NF , of F in M , and such that these neighborhoods are disjoint for different F 's.

Let B_F denote the ϵ -neighborhood of $\{0\} \times F$ in $[0, 1] \times M$. As before, $W := (M \times (0, 1]) \setminus \sqcup_F B_F$ provides a noncompact equivariant cobordism between M and

$\sqcup_F NF$. Our cobordism moment map will now have the form

$$\tilde{\Phi}(m, x) = \Phi(m) + \rho(m, x).$$

We first define ρ locally: let U be an invariant neighborhood of m in M that admits an equivariant retraction to the orbit $T \cdot m$. On the product $U \times (0, 1]$, define

$$\rho_m(m', x) = \alpha_m w(x),$$

where $w : (0, 1] \rightarrow \mathbb{R}$ is such that $w(x)$ approaches infinity as x approaches 0 and vanishes for $\epsilon \leq x \leq 1$, and where $\alpha_m \in \mathfrak{t}^*$ is chosen in the following way. If $m \in M^\eta$ we set $\alpha_m = 0$. If $m \notin M^\eta$ then the vector η is not in the Lie algebra, \mathfrak{g}_m , of the stabilizer of m . We then choose $\alpha_m \in \mathfrak{t}^*$ to be an element of the annihilator, \mathfrak{g}_m^0 , of \mathfrak{g}_m , such that $\langle \alpha, \eta \rangle > 0$.

We patch the functions ρ_m together, using a partition of unity, to obtain a moment map ρ on $W = M \times (0, 1] \sqcup_F B_F$ satisfying $\langle \rho(m, x), \eta \rangle \rightarrow \infty$ as $x \rightarrow 0$ and $\rho(m, x) = 0$ for $\epsilon \leq x \leq 1$.

Finally, take

$$\tilde{\Phi}(m, x) := \Phi(m) + \rho(m, x).$$

This is a moment map on $(M \times (0, 1]) \sqcup_F B_F$. The component $\langle \tilde{\Phi}, \eta \rangle$ is proper, so $\tilde{\Phi}$ itself is proper. On the boundary component $M \times \{1\}$ we get the original moment map:

$$\tilde{\Phi}(m, 1) = \Phi(m).$$

The new boundary component that we get by removing the open set B_F can be identified with an ϵ -neighborhood of F via $(m, x) \mapsto m$, and further identified with the normal bundle NF . Via these identifications, the moment map on this boundary component becomes a moment map Φ'_F on NF that coincides with the map Φ at the points of the zero section (identified with the points of the base F), and whose η -component, $\langle \Phi'_F(v), \eta \rangle$, approaches infinity as v approaches infinity. By Lemma 5.3, this is further cobordant to the moment map Φ_F^η that is given by the formula (6.1).

Finally, the pullback of the equivariant cohomology class c under the projection $M \times (0, 1] \rightarrow M$ restricts to an equivariant cohomology class \tilde{c} on the cobordism manifold W , whose pullback to the boundary component $\partial B_F \cong NF$ is c_F . \square

7. REDUCTION

If (M, ω) is a symplectic manifold with a torus action and Φ is a moment map, the torus acts locally freely on the regular level sets of Φ . The same holds for moment maps in the sense of Definition 2.1; the proof is only slightly harder:

Lemma 7.1. *Let M be a manifold with an action of a torus T , and let Φ be a moment map. If a is a regular value of Φ , the T -action on the level set $\Phi^{-1}(a)$ is locally free, i.e., all stabilizers are discrete.*

Proof. First let us assume that T is a circle. Let F be a component of its fixed point set. Since the moment map is constant on F , the restriction of $d\Phi$ to the tangent bundle of F is zero. Since the moment map is invariant under the circle action, the restriction of $d\Phi$ to the normal bundle of F is zero. This shows that every fixed point for the circle action is a critical point for the moment map.

Now let the torus T be of any dimension. We need show that any point $m \in M$ whose stabilizer has positive dimension is a critical point for Φ . Let H be a circle

subgroup of the stabilizer of m . The previous paragraph, applied to the action of H , implies that m is a critical point for the H -component of Φ , hence for Φ . \square

We denote by $M_{\text{red}} = M_{\text{red}}^\Phi(a)$ the quotient $\Phi^{-1}(a)/T$. By Lemma 7.1, if a is a regular value of Φ , M_{red} is an orbifold. If Φ is proper, M_{red} is compact.

An orientation on M induces an orientation on M_{red} in the following way. At any point $p \in \Phi^{-1}(a)$ we have a (non-canonical) splitting: $T_p M = T_{\Phi(p)} M_{\text{red}} \oplus \mathfrak{t} \oplus \mathfrak{t}^*$ where the \mathfrak{t} -piece is the tangent to the orbit and where the \mathfrak{t}^* -piece is the normal to the level set $\Phi^{-1}(a)$, identified with \mathfrak{t}^* via $d\Phi$. We demand that the orientation of M_{red} , followed by the natural orientation on $\mathfrak{t} \oplus \mathfrak{t}^*$, give the orientation of M .

Consider the inclusion map i and the quotient map π in the following diagram:

$$\begin{array}{ccc} \Phi^{-1}(a) & \xrightarrow{i} & M \\ \pi \downarrow & & \\ M_{\text{red}} & & \end{array} \quad (7.2)$$

An equivariant cohomology class on M descends to an ordinary cohomology class on M_{red} in the following way. An ordinary cohomology class on M_{red} can be viewed as an equivariant cohomology class when M_{red} is taken with the trivial circle action. As such, the class c_{red} on M_{red} that is associated to an equivariant class c on M is characterized by the equality

$$\pi^*(c_{\text{red}}) = i^*(c) \quad (7.3)$$

of equivariant classes on the level set $\Phi^{-1}(a)$. Here we used the fact that if a group acts locally freely on a space, the equivariant cohomology (over \mathbb{R}) is equal to the ordinary cohomology of the quotient.

The map $c \mapsto c_{\text{red}}$ that we have just described is a ring homomorphism from the equivariant cohomology of M to the ordinary cohomology of M_{red} . This is the famous *Kirwan map*.

Definition 7.4. Let M be a manifold with an action of a torus T , let Φ be a moment map on M , and let a be a regular value of Φ . The *reduction* of the triple (M, Φ, c) at the value a is the pair $(M_{\text{red}}, c_{\text{red}})$ where M_{red} is the compact oriented orbifold $\Phi^{-1}(a)/T$ and where c_{red} is the ordinary (real) cohomology class on M_{red} satisfying (7.3).

To avoid ambiguity we may keep track of Φ and a in the notation; we may write $M_{\text{red}}(a)$ and $c_{\text{red}}(a)$, or $M_{\text{red}}^\Phi(a)$ and $c_{\text{red}}^\Phi(a)$.

8. COBORDISM OF REDUCED SPACES

Reduction of a triple (M, Φ, c) , consisting of a manifold with T -action, proper moment map, and equivariant cohomology class, produced a pair $(M_{\text{red}}, c_{\text{red}})$ where M_{red} is a compact orbifold and c_{red} is an ordinary cohomology class on M_{red} .

Definition 8.1. A cobordism between pairs (M, c) and (M', c') , where M and M' are oriented orbifolds and c and c' are cohomology classes, is a pair (W, \tilde{c}) where W is an oriented orbifold-with-boundary and \tilde{c} an ordinary cohomology class on W , and a diffeomorphism of $M \sqcup M'$ with the boundary ∂W , that sends \tilde{c} to $c \sqcup c'$, that respects the orientation on M , and that flips the orientation on M' .

The integral $\int_M c$ is an invariant of cobordism of the pair (M, c) ; this follows from Stokes's theorem.

Lemma 8.2. *Let (M, Φ, c) be a manifold with a T -action, a proper moment map, and an equivariant cohomology class. Let a and b be two values in \mathfrak{t}^* such that all values in the closed interval $[a, b]$ are regular values for Φ . Then $M_{\text{red}}(a)$ is cobordant to $M_{\text{red}}(b)$.*

Proof. The quotient $\Phi^{-1}([a, b])/T$ provides the desired cobordism. The cobording cohomology class is the one whose pullback to $\Phi^{-1}([a, b])$ is the restriction of c . \square

Remark. Under the assumptions of Lemma 8.2, the reduced spaces are diffeomorphic and not just cobordant. (Without the torus action and cohomology class, this follows from Ehresmann's fibration theorem.)

Lemma 8.2 shows that, up to cobordism, the reduced space depends in a robust way on the value at which we reduce. In section 6 we will see that the dependence on the moment map is also robust.

Guillemin was the first to note that “cobordism commutes with reduction” is true and is useful for symplectic manifolds. Here is the statement for moment maps in our new sense:

Lemma 8.3. *Let (M, Φ, c) and (M', Φ', c') be cobordant oriented manifolds with T -actions, proper moment maps, and equivariant cohomology classes. Let $a \in \mathfrak{t}^*$ be a value that is regular for both Φ and Φ' . Then the corresponding reductions, $(M_{\text{red}}, c_{\text{red}})$ and $(M'_{\text{red}}, c'_{\text{red}})$, are also cobordant.*

Proof. Let $(W, \tilde{\Phi}, \tilde{c})$ be a cobording manifold, proper moment map, and equivariant cohomology class. Since Φ and Φ' are proper, their sets of regular values are open, so any value b that is close enough to a is also a regular value for Φ and for Φ' . For such b , the reduced spaces at b are cobordant to the reduced spaces at a , by Lemma 8.2. If we choose such a b that is also a regular value for $\tilde{\Phi}$, the quotient $\tilde{\Phi}^{-1}(b)/T$ provides a cobordism between the reduced spaces at b . The cobording cohomology class is the one whose pullback to $\tilde{\Phi}^{-1}(b)$ is the restriction of \tilde{c} . \square

Corollary 8.4. For every regular value $a \in \mathfrak{t}^*$, the integral $\int_{M_{\text{red}}(a)} c_{\text{red}}(a)$ is an invariant of cobordism of the triple (M, Φ, c) .

9. HAMILTONIAN MANIFOLDS

Corollary 8.4 implies that the Duistermaat-Heckman measure of a Hamiltonian T -manifold is an invariant of cobordism. We first recall some definitions:

Definition 9.1. Let (M, ω) and (M', ω') be two symplectic manifolds acted upon by a torus T with proper moment maps Φ and Φ' . A *cobordism* between them is an oriented manifold-with-boundary W with a T -action, a closed two-form $\tilde{\omega}$, and a proper moment map $\tilde{\Phi} : W \rightarrow \mathfrak{t}^*$ (i.e., a map satisfying $d\langle \tilde{\Phi}, \xi \rangle = \iota(\xi_W)\tilde{\omega}$ for all $\xi \in \mathfrak{t}$, when ξ_W is the vector field on W corresponding to ξ), and an equivariant diffeomorphism of $M \sqcup M'$ with the boundary ∂W that carries $\tilde{\omega}$ to $\omega \sqcup \omega'$ and $\tilde{\Phi}$ to $\Phi \sqcup \Phi'$, and that respects orientation on M and flips orientation on M' .

Remark. In the above definition there is no reason other than psychological to assume that the ω and ω' are nondegenerate. They can be just closed two-forms. In fact, any two closed two-forms on the same manifold that belong to the same cohomology class are cobordant, so we can even work with cohomology classes instead of with two-forms.

Definition 9.2. *Liouville measure* on a symplectic manifold (M, ω) is defined on open sets by integrating the volume form $\omega^n/n!$ ($n = \dim M$) with respect to the symplectic orientation. In the presence of a T -action, the corresponding *Duistermaat-Heckman measure* is the measure on \mathfrak{t}^* obtained as the push-forward of Liouville measure by the moment map.

Let the torus T act on a symplectic manifold (M, ω) with a proper moment map Φ . At any regular value a , the Marsden-Weinstein reduced space is the compact symplectic orbifold $(M_{\text{red}}, \omega_{\text{red}})$ defined by $M_{\text{red}} = \Phi^{-1}(a)/T$ and $\pi^* \omega_{\text{red}} = i^* \omega$, where i and π are the inclusion and quotient maps as in (7.2).

Definition 9.3. The *Duistermaat-Heckman function* associates to a regular value of the moment map the symplectic volume of the reduced space: $a \mapsto \int_{M_{\text{red}}} \omega_{\text{red}}^d / d!$, $M_{\text{red}} = \Phi^{-1}(a)/T$, $d = \dim M_{\text{red}}$.

The Duistermaat-Heckman measure is absolutely continuous with respect to Lebesgue measure on \mathfrak{t}^* , provided that the torus action is effective. It is well known, and is not hard to prove, that the Radon-Nikodym derivative of the Duistermaat-Heckman measure is the Duistermaat-Heckman function.

The Kirwan map (see §7) sends the equivariant cohomology class $[\omega + \Phi]$ on M to the ordinary cohomology class $[\omega_{\text{red}} + a]$ (of mixed degree) on $M_{\text{red}}(a)$.

Finally, Corollary 8.4 applied to the equivariant cohomology classes of $(\omega + \Phi - a)^d / d!$ on M implies the following:

Proposition 9.4. *The Duistermaat-Heckman measure is an invariant of cobordism.*

Remark. The fact that the Duistermaat-Heckman measure is invariant under (compact) cobordisms was observed by V. Ginzburg several years ago. This was one of the main motivations for introducing cobordism techniques into equivariant symplectic geometry.

10. APPLICATION: THE JEFFREY-KIRWAN LOCALIZATION FORMULA

Theorem 1, together with the facts that “cobordism commutes with reduction” (Lemma 8.3) and that the integral of a reduced class is an invariant of cobordism (Corollary 8.4), yields a formula for integrals over the reduced space in terms of fixed point data:

Proposition 10.1. *Let M be an oriented manifold with an action of a torus, T . Let Φ be a moment map, and let c be an equivariant cohomology class on M . Suppose that the vector $\eta \in \mathfrak{t}$ is such that the η -component of the moment map,*

$$\langle \Phi, \eta \rangle : M \longrightarrow \mathbb{R},$$

is proper and bounded from below. Denote by M^η the zero set of the vector field on M corresponding to η . For every connected component, F , of M^η , denote by NF the normal bundle of F in M , and let $\Phi_F^\eta : NF \longrightarrow \mathfrak{t}^$ the proper moment described in Formula (6.1). Let c_F be the equivariant cohomology class on NF obtained by pulling back c via the inclusion $F \hookrightarrow M$ and further via the bundle map $NF \longrightarrow F$. Let $a \in \mathfrak{t}^*$ be a value that is regular for Φ and for all the Φ_F^η 's. Let $M_{\text{red}} = M_{\text{red}}^\Phi(a)$ and $(NF)_{\text{red}} = (NF)_{\text{red}}^{\Phi_F^\eta}(a)$ be the corresponding reduced spaces,*

and let $c_{\text{red}} = c_{\text{red}}^{\Phi}(a)$ and $(c_F)_{\text{red}} = (c_F)_{\text{red}}^{\Phi_F^{\eta}}(a)$ be the corresponding cohomology classes on these reduced spaces. Then

$$\int_{M_{\text{red}}} c_{\text{red}} = \sum_F \int_{NF_{\text{red}}} (c_F)_{\text{red}}. \quad (10.2)$$

Notice that in (10.2), if we choose η generic, the sum is over the components of the fixed point set, M^T . Also notice that the F -summand is nonzero if and only if $\langle \Phi(F), \eta \rangle < 0$.

Proposition 10.1 generalizes Guillemin's topological form for the Jeffrey-Kirwan localization theorem in the abelian case. Jeffrey and Kirwan consider Hamiltonian actions of compact nonabelian groups on compact symplectic manifolds. Every cohomology class on the reduced space at 0 comes via "Kirwan's map" from an equivariant cohomology class on the manifold (as in (7.3)). Jeffrey and Kirwan express the integral of such a class over the reduced space by an explicit formula that only depends on data at the set of points that are fixed under the action of a maximal torus. Shaun Martin showed, by topological means, that their formula would follow from a localization formula for abelian groups. Guillemin interpreted this abelian localization formula as Formula (10.2).

11. APPLICATION: THE GUILLEMIN-LERMAN-STERNBERG FORMULA

A special case of Proposition 10.1 is the formula of Guillemin, Lerman, and Sternberg for the Duistermaat-Heckman measure. (The definition of the Duistermaat-Heckman measure was recalled in §9.)

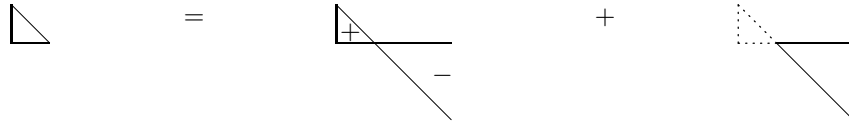
Proposition 11.1. *Let (M, ω) be a presymplectic oriented manifold with a Hamiltonian action of the torus T and with a moment map $\Phi : M \rightarrow \mathfrak{t}^*$ (in the traditional sense: $d\langle \Phi, \xi \rangle = -\iota(\xi_M)\omega$ for all $\xi \in \mathfrak{t}$, where ξ_M is the corresponding vector field on M). Let $\eta \in \mathfrak{t}$ be a vector such that the function $\langle \Phi, \eta \rangle : M \rightarrow \mathbb{R}$ is proper and bounded from below. Let M^{η} be the set of zeros of the vector field on M that corresponds to η .*

For every component F of the set M^{η} , let NF be the normal bundle of F in M , choose an invariant metric on the fibers of NF , and let Φ_F^{η} be the moment map described in Theorem 1. There exists a symplectic form on the fibers of NF for which Φ_F^{η} , restricted to each fiber, is a moment map in the usual, symplectic, sense. With these forms NF becomes a symplectic vector bundle. Let ω_F be any closed two-form on the total space of the bundle NF whose restriction to the fibers is given by the symplectic forms just mentioned, and whose pullback to F via the zero section coincides with the pullback of ω under the inclusion of F into M . (Such a closed two-form, ω_F , can be constructed by "minimal coupling"; see, e.g., [GLS2]. Notice that ω_F need not be symplectic on NF .) Then $\Phi_F^{\eta} : NF \rightarrow \mathfrak{t}^$ is a moment map for (NF, ω_F) .*

Denote by $DH(M)$ the Duistermaat-Heckman measure corresponding to the triple (M, ω, Φ) , and by $DH(NF)$ the Duistermaat-Heckman measure corresponding to the triple (NF, ω_F, Φ_F) . Then

$$DH(M) = \sum_F DH(NF). \quad (11.2)$$

The Guillemin-Lerman-Sternberg formula was originally asserted for compact symplectic manifolds and for torus actions with isolated fixed points. The vector

FIGURE 1. The Guillemin-Lerman-Sternberg formula for \mathbb{CP}^2 FIGURE 2. The (generalized) Guillemin-Lerman-Sternberg formula for \mathbb{CP}^2 , with a non-generic choice of “polarizing vector”

$\eta \in \mathfrak{t}$ was chosen to be generic, in the sense that the zero set M^η was required to be equal to M^T , the fixed point set for T in M . The right hand side of (11.2) in this case is the sum of the Duistermaat-Heckman measures, $\text{DH}(T_p M)$, for the linear torus actions on the tangent spaces at the fixed point. The original formula of Guillemin-Lerman-Sternberg [GLS1] involved an explicit formula for the measure $\text{DH}(T_p M)$ as the push-forward of Lebesgue measure on a positive orthant in \mathbb{R}^n , $n = \frac{1}{2} \dim M$, via a linear projection to \mathfrak{t}^* . See [GLS2].

Example 11.3. Take \mathbb{CP}^2 with the Fubini-Study symplectic form, whose pullback to $S^5 \subset \mathbb{C}^3$ is the standard two-form $\sum_{j=1}^3 dx_j \wedge dy_j$. The linear action of $T = S^1 \times S^1$ on \mathbb{C}^3 by scalar multiplication on the first two coordinates induces an action on \mathbb{CP}^2 . The image of the moment map is Lebesgue measure on a triangle. The action on \mathbb{CP}^2 has three isolated fixed points; their images are the vertices of the triangle. Each of the summands $\text{DH}(T_p M)$ is Lebesgue measure on the region between two rays and zero outside. The Guillemin-Lerman-Sternberg formula exhibits the triangle as a combination of three such “wedges”; see figure 1. Notice that this involves a choice of direction in which the infinite rays are pointing; this is the choice of the vector η .

Example 11.4. In Example 11.3, if η is chosen non-generically, the zero set M^η consists of one copy of \mathbb{CP}^1 and one isolated fixed point. The (generalized) Guillemin-Lerman-Sternberg formula then gives the combination illustrated in Figure 2. Here the first summand is the Duistermaat-Heckman measure for the normal bundle of \mathbb{CP}^1 in \mathbb{CP}^2 . This summand is a *signed* measure; this reflects the fact that the cohomology class on this normal bundle does not have an invariant symplectic representative.

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